Thin Cylindrical Shells under Surface Torques

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Introduction

In a previous paper[1] the bending problem of an infinitely long cylinder loaded with concentrated equal and opposite forces, acting at the ends of a vertical diameter, was discussed. The loading function in this case was presented by a Fourier integral in the longitudinal direction and by a Fourier series in circumferential direction. The integral representation has the advantage that the boundary conditions are automatically taken care of, and no subsequent determination of Fourier coefficients is necessary. The Fourier coefficients and the undetermined function in the Fourier integral are determined simply from the loading condition. The radial displacement was then obtained from the solution of the eighth-order differential equation.

In the present paper, the shearing stress-resultant of an infinitely long cylindrical shell subjected to two equal and opposite torques acting about the radial axis on the surface of the shell, as shown in Fig. 1a, has been analyzed. The solution of this problem was achieved by replacing the torque with two equal and opposite forces acting about the same axis at an infinitely small distance apart. The moment produced by the pair of forces is of the same magnitude and sign as the torque. In the case of a plate, the shearing stresses produced by two equal and opposite forces acting perpendicularly to either axis are identical.[2] However, in the present problem, the phenomenon is quite different because of the effect of the curvature of the shell on the displacements. For this reason, in the present investigation, a combination of two pairs of forces with moments are introduced to replace the applied torque. The forces are so arranged that one pair of forces, equal and opposite in magnitude, act at an infinitely small distance apart in the direction of x axis and the other pair of forces under the same condition in the direction of s axis. The loading distribution functions under consideration may be presented by a combination of a Fourier series and a Fourier integral along the circumference and the generatrix, respectively.

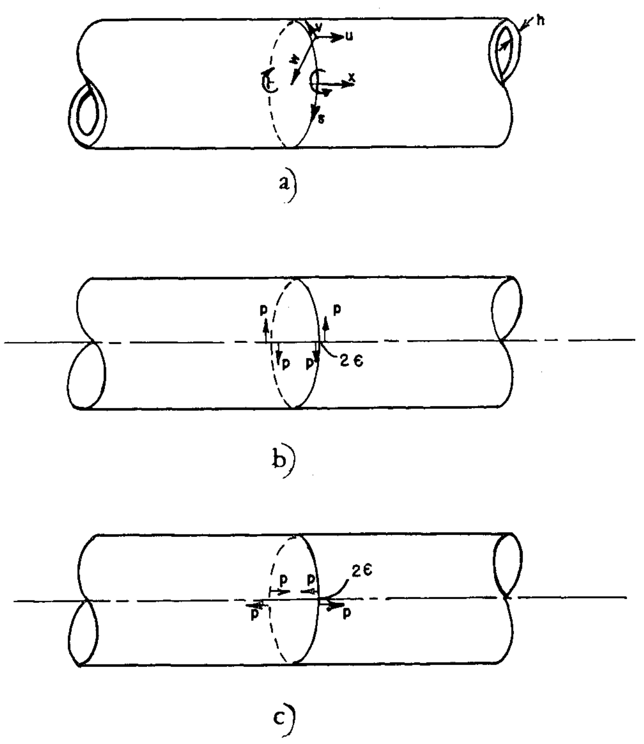

Fig. 1 Applied torque acting on an infinitely long cylinder.

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Fundamental Equations

The equations of equilibrium of a shell element under a circumferential pressure or a longitudinal pressure of an intensity q, as shown in Fig. 1b and Fig. 1c, respectively, can be derived from considerations of the equilibrium of forces and moments. In terms of the displacements, the equilibrium equations can be written in the following simplified form:

$$\nabla^4 u - \frac{1}{a}\left[\nu \frac{\partial^3 w}{\partial x^3} - \frac{\partial^3 w}{\partial x \partial s^2}\right] + \frac{(1+\nu)}{12(1-\nu)}\frac{h^2}{D} f_1(q) = 0 \quad (1)$$

$$\nabla^4 v - \frac{1}{a}\left[\frac{\partial^3 w}{\partial s^3} + (2-\nu)\frac{\partial^3 w}{\partial s \partial x^2}\right] - \frac{1}{12(1-\nu)}\frac{h^2}{D} f_2(q) = 0 \quad (2)$$

$$\nabla^8 w + \frac{12(1-\nu^2)}{a^2 h^2}\frac{\partial^4 w}{\partial x^4} - \frac{1}{Da} f_3(q) = 0 \quad (3)$$

where

$$f_1(q) = (1+\nu)\frac{\partial^2 q}{\partial x \partial s} \qquad f_2(q) = 2\frac{\partial^2 q}{\partial x^2} + (1-\nu)\frac{\partial^2 q}{\partial s^2}$$

$$f_3(q) = (2+\nu)\frac{\partial^3 q}{\partial x^2 \partial s} + \frac{\partial^3 q}{\partial s^3}$$

for load acting tangentially and

$$f_1(q) = 2\frac{\partial^2 q}{\partial s^2} + (1-\nu)\frac{\partial^2 q}{\partial x^2} \qquad f_2(q) = (1+\nu)\frac{\partial^2 q}{\partial x \partial s}$$

$$f_3(q) = \frac{\partial^3 q}{\partial x \partial s^2} - \nu\frac{\partial^3 q}{\partial x^3}$$

for load acting longitudinally.

In the foregoing equations, h is the thickness, a the mean radius of the shell, ν the Poisson's ratio, and $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the shell.

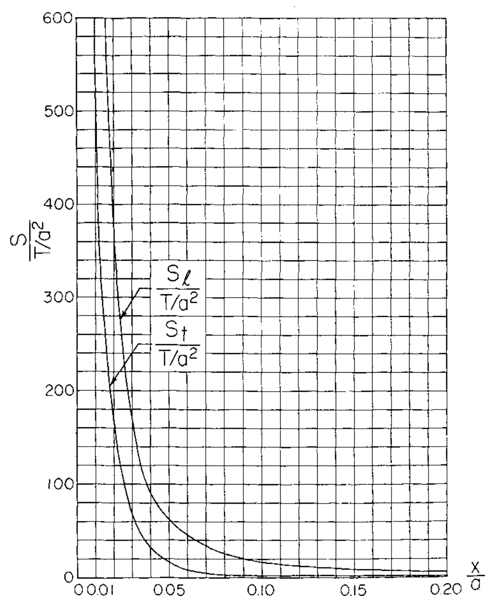

Fig. 2 Shearing stress-resultants distribution along the generatrix ($0.01 < x/a < 0.20$).

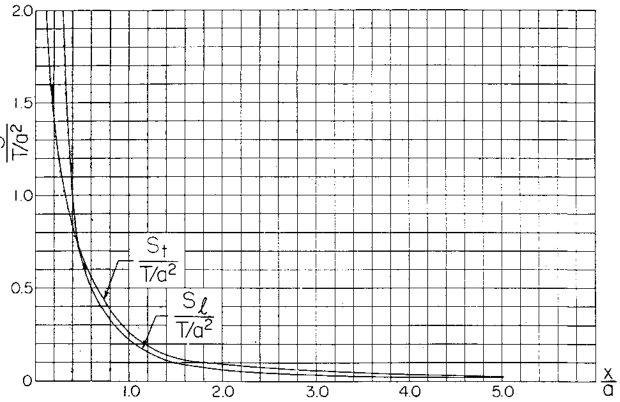

Fig. 3 Shearing stress-resultants distribution along the generatrix ($0.20 < x/a < 5.0$).

It is interesting to note that Eq. (3) becomes the well-known differential equation of a flat plate if the radius of the shell a is made infinitely large. In this case the lateral deflection of an infinitely long plate is evidently equal to zero.

Determination of Shearing Stress Distribution

The shearing stress-resultant in the wall of a cylindrical shell due to two equal and opposite surface torques in the case of loads acting tangentially (Fig. 1b) is given as[3]

$$S_t = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}\right) \tag{4}$$

where the displacements u and v can be determined from Eqs. (1) and (2).

The load distribution under consideration may be represented by an even function along the circumference and by an odd function along the generatrix. As discussed in Ref. 1, the load distribution function can be expressed by a combination of Fourier series and a Fourier integral

$$q(x, s) = \left[\frac{q_0}{2} + \sum_{n=2,4\ldots}^{\infty} q_n \cos\frac{ns}{a}\right]\int_0^{\infty} f(\lambda)\sin\frac{\lambda x}{a}\,d\lambda \tag{5}$$

The three components of displacement, u, v, w, can be expanded in a similar manner in terms of three undetermined functions, $u(\lambda)$, $v(\lambda)$, and $w(\lambda)$, respectively,

$$u = \sum_{n=0,2\ldots}^{\infty} \sin\frac{ns}{a}\int_0^{\infty} u(\lambda)\cos\frac{\lambda x}{a}\,d\lambda \tag{6}$$

$$v = \sum_{n=0,2\ldots}^{\infty} \cos\frac{ns}{a}\int_0^{\infty} v(\lambda)\sin\frac{\lambda x}{a}\,d\lambda \tag{7}$$

$$w = \sum_{n=0,2\ldots}^{\infty} \sin\frac{ns}{a}\int_0^{\infty} w(\lambda)\sin\frac{\lambda x}{a}\,d\lambda \tag{8}$$

The function q_n and $f(\lambda)$ in Eq. (5) can be determined by developing them from the loading condition. The transformed Fourier integral[4] of Eq. (5), which represents the loading condition along the longitudinal direction, is given by

$$f(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} q\left(\frac{x}{a}\right)\sin\frac{\lambda x}{a}\,d\left(\frac{x}{a}\right) = \frac{4}{\pi\lambda}\sin^2\frac{\lambda\epsilon}{a} \tag{9}$$

where $q(x/a) = \pm 1$ when $0 \leqq x \leqq 2\epsilon$ and $0 \geqq x \geqq -2\epsilon$, respectively, and $q(x/a) = 0$ when $x > 2\epsilon$ and $x < -2\epsilon$. Similarly, q_n can be determined from the expansion of loading function along the circumference in a Fourier series:

$$q_n = \frac{2}{\pi a}\int_{-\pi a/2}^{\pi a/2} q\left(\frac{s}{a}\right)\cos\frac{ns}{a}\,ds = \frac{4q}{n\pi}\sin\frac{nc}{a} \tag{10}$$

where $q(s/a) = q$ when $c \geqq s \geqq 0$ and $-c \leqq s \leqq 0$, and $q(s/a) = 0$ when $c < s \leqq \pi a/2$ and $-c > s \geqq -\pi a/2$.

Next, the case of a concentration torque applied at the origin may be considered. Such a load can be obtained by making the length 2ϵ and $2c$ of the loaded portion infinitely small as follows:

$$\text{torque} = T = P(2\epsilon) = 8q\,\epsilon^2 c \tag{11}$$

Now Eqs. (1) and (2) can be solved with the aid of Eqs. (5–11). Substituting the results of $\partial u/\partial s$ and $\partial v/\partial x$ in Eq. (4), the shearing stress-resultant is obtained as follows:

$$\begin{aligned}
\frac{S_t}{T/a^2} = {} & \frac{1}{8\pi}\sum_{n=2,4\ldots}^{\infty}\frac{\cos(ns/a)}{(1+j^2)^{1/2}}\times \\
& \Bigg\{\Bigg[(2+\nu)\left(\frac{(2J)^{1/2}}{n}[(1+j^2)^{1/2}-j]^{1/2}\right) + \\
& (1+\nu)\Bigg(\frac{2\cdot 2^{1/2}n}{J^{1/2}}[(1+j^2)^{1/2}+j]^{1/2} - \\
& \frac{2(1+j^2)^{1/2}\phi}{n^2} + \frac{J^{3/2}}{2\cdot 2^{1/2}n^3}[(1+j^2)^{1/2}+j]^{1/2}\Bigg)\Bigg]\times \\
& \Bigg[\left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right)e^{-B(x/a)} + \\
& \left(C\cos\frac{Cx}{a} - G\sin\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg] + \\
& 4(1+j^2)^{1/2}\Bigg[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)e^{-B(x/a)} + \\
& \left(C\cos\frac{Cx}{a} + G\cos\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg] - \\
& \Bigg[2(1+\nu)\left(\frac{2}{J}\right)^{1/2} n\,[(1+j^2)^{1/2}-j]^{1/2} - \\
& \frac{(2J)^{1/2}}{n}[(1+j^2)^{1/2}+j]^{1/2}\Bigg]\Bigg[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)\times \\
& e^{-B(x/a)} - \left(G\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg]\Bigg\}
\end{aligned} \tag{12}$$

where $J^2 = 12(1-\nu^2)\,(a/h)^2$, $j = J/4n^2$, and the constants A, B, C, and G are given in Ref. 1.

In a similar manner as in the case of loads acting tangentially, the shearing stress-resultant S_l due to two equal and opposite surface torques in the case of loads acting longitudinally (Fig. 1c) is obtained.

$$\begin{aligned}
\frac{S_l}{T/a^2} = {} & \frac{1}{2\pi}\sum_{n=1,3\ldots}^{\infty}\cos\frac{ns}{a}\times \\
& \Bigg[\Bigg[\left\{(1+\nu)n - (1+\nu)n^2\frac{x}{a}\right\}e^{-n(x/a)} - \\
& \frac{1}{4J(1+j^2)^{1/2}}\Bigg\{\Bigg[\left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right)e^{-B(x/a)} + \\
& \left(C\cos\frac{Cx}{a} - G\sin\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg]\times \\
& \left[\frac{2\nu J\eta}{n^2} + (1+\nu)\left(4\phi - \frac{2R_2\phi}{n^4} + \frac{J^2\phi}{2n^4}\right)\right] - \\
& \Bigg[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)e^{-B(x/a)} - \\
& \left(D\cos\frac{Cx}{a} + \mathrm{C}\sin\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg]\left[4(1+\nu)\eta + \frac{2J\phi}{n^2}\right] - \\
& \frac{4R_2}{n^2}\Bigg[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)e^{-B(x/a)} + \\
& \left(G\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right)e^{-G(x/a)}\Bigg]\Bigg\}\Bigg]
\end{aligned} \tag{13}$$

The shearing stress-resultants S_t and S_l of an infinitely long cylindrical shell subjected to two equal and opposite torques acting about the radial axis on the surface of the shell are given in Eqs. (12) and (13), respectively. It can be seen that the expressions for the shearing stress-resultants satisfy the following conditions: at the point where the torque is applied the stress-resultant is maximum, and it vanishes at infinity.

The results obtained in Eqs. (12) and (13) may be used to verify that the total torque produced by the force-resultants is equal to the applied torque. The calculation of the total torque can be made by summing up all the force-resultants multiplied by their corresponding moment arms of an infinitesimal element in the immediate vicinity of the applied torque. The normal stress-resultants acting on the element in both x and s directions are obtained from the equilibrium condition in terms of the corresponding shearing stress-resultants. The result proved to be correct.

Figures 2 and 3 show the shearing stress-resultants distribution along the generatrix (between $x = h$ and $x = 500h$) of an infinitely long cylinder with $a/h = 100$. The computation of the shearing stess-resultants was made with the aid of a CDC 1604 digital computer. It was noted that the shearing stress-resultant near the origin due to the applied torque composed of a pair of forces in the x direction is much larger than that due to a pair of forces in the s direction. The reason for the discrepancy is due to the effect of the curvature of the shell on the displacements. The results also show that the shearing stress-resultants decrease very rapidly from the region near the applied torque outward along the generatrix.

References

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[2] Love, A. E. H., *The Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), 4th ed.

[3] Timoshenko, S. P., *Theory of Plates and Shells* (McGraw-Hill Book Co., Inc., New York, 1940), 1st ed.

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Natural Frequency Curves of Simply Supported Cylindrical Shells

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The problem of determining the natural frequency of cylindrical shells has been studied by many authors. A brief historical review may be found in Ref. 1. So far, the frequency curves obtained by the former investigators do not cover the ranges of wave number and shell radius-thickness ratio broad enough for practical use.

In this brief note, a set of frequency curves is presented. They cover the radius-thickness ratio from 100 to 1500, the circumferential wave number from 0 to 38, and axial wave number ma π/l from 0.3 to 30.

The equation of motion used was obtained in the manner of Morley[2]:

$$\nabla^4(\nabla^2 + 1)^2 w + 4K^4 \frac{\partial^4 w}{\partial x^4} + \frac{a^4}{D}\rho h \nabla^4 \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

Here w is the quantity of the radial displacement w' divided by the shell radius a. x' and θ are the cylindrical coordinate system and $x = x'/a$:

$$\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial \theta^2)$$

$$D = Eh^3/12(1 - \nu^2)$$

$$K^4 = 3(1 - \nu^2)(a/h)^2$$

E and ν are, respectively, Young's modulus and Poisson's ratio, and ρ and h are, respectively, the mass density of the material and the thickness of the shell.

For a simply supported cylindrical shell in free vibration, the radial motion is taken as

$$w = e^{i\omega t} \times C_{sn} \cos sx \times \cos n\theta \qquad (2)$$

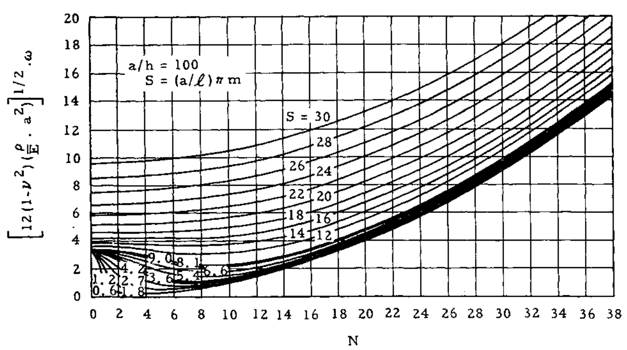

Fig. 1 Natural frequency of cylindrical shells.

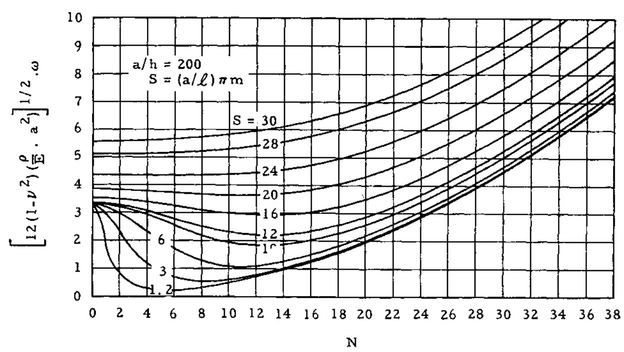

Fig. 2 Natural frequency of cylindrical shells.

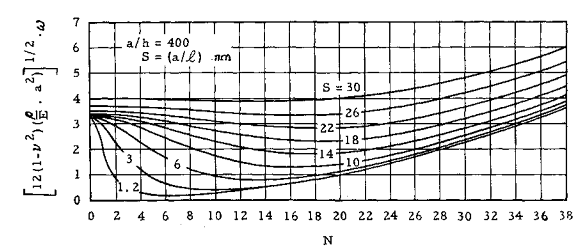

Fig. 3 Natural frequency of cylindrical shell.

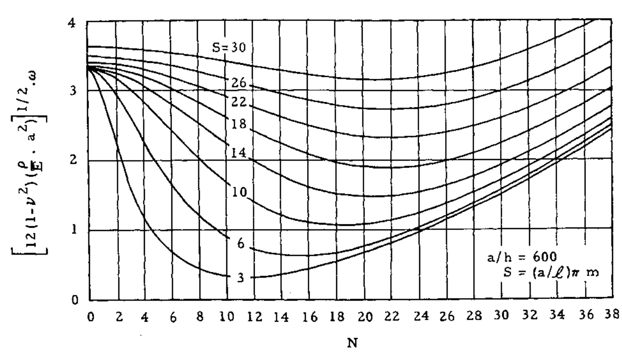

Fig. 4 Natural frequency of cylindrical shell.

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